Convection without the Mixing Length Parameter

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Stefano Pasetto and Mark Cropper
Mullard Space Science Lab, University College London

+ Cesare Chiosi, Emanuela Chosi, Achim Weiss, Eva Grebel
Rationale for replacing Mixing Length Theory

• The current approach for convection is Mixing Length Theory [Prandtl (1925), Böhm–Vitense (1958)]

• The universal applicability of the MLT is unproven and requires a calibration for each star
  ⇒ a self-consistent theory will be a significant advance (and overdue)

• The correct treatment of convection is critical for stellar models throughout the H-R diagram
  ⇒ affects every aspect of stellar and galactic evolution

• Advances in asteroseismology have allowed the internal structure of stars to be measured directly with increasing accuracy
  ⇒ allows detailed confrontation with stellar models

• Advent of scale and accuracy of Gaia data requires stellar models of greater fidelity to fully utilise it
  e.g. location of red giant tracks depends sensitively on MLT parameter
Convection Theory: stability criteria

- Energy transfer by convection in the classical treatment is a linear "stability study" against non-spherical perturbations.

Assuming that $dr$ is small and $p_{\text{star}}+dp_{\text{star}} = p_{\text{sur}}+dp_{\text{sur}}$ leads to the Schwarzschild/Ledoux criterion for instability, i.e. convection.

$p_{\text{sur}}$ is the pressure at the surface of the bubble.

Figure 5.3. Schematic illustration of the Schwarzschild criterion for spherical perturbations.
Mixing Length Theory

- The formulation is set in terms of:
  \( \varphi_{\text{rad}} \) the radiative energy flux
  \( \varphi_{\text{conv}} \) the convective energy flux
  \( \nabla \) the stellar temperature gradient with respect to pressure
  \( \nabla_e \) the element temperature gradient with respect to pressure
  \( d \log T/d \log P \)
  \( \left| d \ln T/d \ln P \right|_e \)

- With the assumption that \( l_m \equiv \Lambda_m h_P \) where
  \( l_m \) is the mean free path of a convective element
  \( h_P \) is the distance scale of the pressure stratification
  \( \Lambda_m \) is the proportionality constant (the Mixing Length Parameter)

the system of equations

\[
\begin{align*}
\varphi_{\text{rad|cnd}} & = \frac{4ac}{3} \frac{T^4}{\kappa h_P \rho} \nabla \\
\varphi_{\text{rad|cnd}} + \varphi_{\text{conv}} & = \frac{4ac}{3} \frac{T^4}{\kappa h_P \rho} \nabla_{\text{rad}} \\
\bar{v}^2 & = g \delta (\nabla - \nabla_e) \frac{i_m^2}{8h_P} \\
\varphi_{\text{conv}} & = \rho c_P T \sqrt{g \delta} \frac{i_m^2}{4\sqrt{2}} h_P^{-3/2} (\nabla - \nabla_e)^{3/2} \\
\frac{\nabla_e - \nabla_{\text{ad}}}{\nabla - \nabla_e} & = \frac{6acT^3}{\kappa \rho^2 c_P l_m \bar{v}},
\end{align*}
\]

can be solved.
A self-consistent theory: two papers

  - Paper 1 formulates the problem in the reference frame of the moving convective element
  - This allows the identification of a self-consistent additional constraint which can be used to close the system of equations without the external imposition of a mixing-length parameter
  - A comparison is made of the derived parameters (e.g., sound speed) in the Sun (where the Mixing Length Theory is calibrated)

  - Paper 2 presents the first stellar models using the non-MLT treatment
  - Evolutionary tracks are derived and compared to MLT-derived tracks
  - Derived internal parameters are compared between the two theories and agreement is found to be satisfactory
Self-consistent Theory: stability criterion

- The new treatment is in the co-moving frame of the bubble

co-moving coordinates + the concept of the “velocity potential”

- The instability criterion now translates to a criterion that

\[ \frac{v}{\dot{\xi}_e} \ll 1 \]

i.e. the new instability criterion is a velocity criterion that the expansion speed of the bubble is greater than the speed of the bubble in the star
Relation between blob size and time

- the **unstable** expansion is in terms of hyper-geometric functions which is **quadratic** in time in the leading term

\[ \tau = \frac{t}{t_0} \]

(normalised time)
Formulation

- Pasetto et al (2014) derives 6 equations in 6 unknowns:

\[
\begin{align*}
\varphi_{\text{rad/cnd}} &= \frac{4ac}{3} \frac{T^4}{\kappa h_P \rho} \nabla \\
\varphi_{\text{rad/cnd}} + \varphi_{\text{cnv}} &= \frac{4ac}{3} \frac{T^4}{\kappa h_P \rho} \nabla \varphi_{\text{rad}} \\
\bar{v}^2 &= \frac{3h_P}{2\delta \bar{v} \tau} + \left( \nabla \varphi_{\text{rad}} + 2 \nabla - \frac{\varphi}{2\delta} \nabla \mu \right) \xi e g \\
\varphi_{\text{cnv}} &= \rho c_P T (\nabla - \nabla \varphi_{\text{rad}}) \frac{\bar{v}^2 \tau}{h_P} \\
\frac{\nabla \varphi_{\text{cnv}}}{\nabla - \nabla \varphi_{\text{ad}}} &= \frac{4ac T^3}{\kappa \rho^2 c_P} \frac{\tau}{\xi e^2} \\
\bar{\xi} &= \frac{g}{4} \frac{3h_P}{2\delta \bar{v} \tau} + \left( \nabla \varphi_{\text{rad}} + 2 \nabla - \frac{\varphi}{2\delta} \nabla \mu \right) \bar{\chi},
\end{align*}
\]

- The two new unknowns are:
  \(\bar{\xi}\) the mean size of the convective element and
  \(\bar{v}\) the mean velocity
Solving the system of equations

- After substitutions and definition of new variables, the 6 equations reduce to the following:

\[
\frac{Y^2}{(W - \eta)(\eta - Y)} = \frac{1}{3} \frac{\bar{\chi}}{\tau^2}
\]

where:

- \(W \equiv \nabla_{\text{rad}} - \nabla_{\text{ad}} > 0\),
- \(\eta \equiv \nabla - \nabla_{\text{ad}}\),
- \(Y \equiv \nabla - \nabla_{e}\)
- \(\chi \equiv \frac{\xi_e}{\xi_0}\)

- but, recall \(\chi \equiv \frac{\xi_e}{\xi_0} \propto \tau^2\) from previous graph, so \(\frac{\bar{\chi}}{\tau^2} = \text{constant}\)
Outcome of the reduction of dimensionality

- The new system of equations has a new invariant manifold on which all the solutions live.

- The temperature gradients at each point in any star are located on this manifold.

- "Theorem of Unicity": a relation between the 4 fundamental gradients that govern the energy transfer inside a star.
Another important consequence

- The treatment leads to a non-hydrostatic equilibrium theory, hence non-hydrostatic equilibrium models of atmospheres

- This is a fundamental advance on the MLT where equilibrium is assumed to be reached at the end of the bubble movement

\[\chi \equiv \frac{\xi_e}{\xi_0}\] normalised blob size

Y axis is the pressure difference across the blob interface compared to pressure in the same stellar layer far from the blob
### Results (1): outer convective layers

**Solar Model**
Bertelli et al. (2008)

black: MLT ($\Lambda = 1.68$)
red: this work

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**Expanded pressure scale**
Results (2): outer convective layers

\[ \log L / L_{\odot} = 2.598, \log T_{\text{eff}} = 3.593 \]

\textit{Bertelli et al. (2008)}

\[ \text{black: MLT (} \Lambda = 1.68 \text{)} \]

\[ \text{red: this work} \]

\[ \text{2M}_{\odot} \text{ RGB star} \]
Outer convective layers: comparison

- For Solar model:
  - good agreement for convective and radiative fluxes throughout
  - temperature gradients are in good agreement except for surface layers
    Reason: treatment incomplete at the boundary

- For $2M_{\odot}$ model:
  - good agreement for convective fluxes
  - divergence to lower boundary for radiative fluxes
    Reason: these solutions are not constrained to match the inner solution at the transition layer
  - temperature gradients as for Solar model

- To constrain the inner solution, need to calculate full stellar models

- For these full calculations, Mixing Length Theory used for the interiors
Results 3: Stellar models

Figure 9. The HRD of the 0.8, 1.0, 1.5, 2.0, and 2.5 $M_{\odot}$ stars with initial chemical composition $X=0.703, Y=0.280, Z=0.017$ calculated from the main sequence to advanced evolutionary stages using both the classical ML theory (the crossed lines) and the SFC theory (dotted lines of different colors). The 0.8, 1.0, and 2.0 $M_{\odot}$ models are carried to a late stage of the RGB before core He-ignition (He-Flash), whereas the 2.5 $M_{\odot}$ is evolved up to very advanced stages of central He-burning ($Y_c \approx 0.1$). The stellar models are calculated with the Padova code and input physics used by Bertelli et al. (1994) and Bertelli et al. (2008), see also the text for more details. The models are meant to prove doubt that the SFC theory with no ML parameters is equivalent to the classical ML theory with calibrated ML parameter ($\Lambda_m = 1.68$ in our case).

With Magic et al. (2015) of 3D radiative hydrodynamic simulations of convection in the envelopes of late-type stars in terms of the 1D classical ML theory. Using different calibrators and mapping the results as function of gravity, effective and effective temperature Magic et al. (2015) find that at given gravity the ML parameter increases with decreasing effective temperature, the opposite at given effective temperature and decreasing gravity. There are also additional dependencies on metallicity and stellar mass that we leave aside here. Looking at the case of the Sun, passing from the main sequence to a late stage on the RGB, the ML is found to decrease by as much as about 10 percent. Applying this to stellar models, a less steep RGB would result as shown by our model calculations with the SFC theory. Owing to the complexity of the new SFC theory with respect to the classical ML theory, the results are very promising. These preliminary model calculations show that the SFC theory with no ML parameter is equivalent to the classical ML theory with calibrated ML. More work is necessary to establish a quantitative correspondence between the two theories of convection.

CONCLUSIONS AND FUTURE WORK

We have presented here the first results of the integration of stellar atmospheres and exploratory full stellar models to which the new convection theory developed by Pasetto et al. (2014) has been applied. To this aim, a mathematical and computational algorithm and a companion code have been developed to integrate the system of equations governing the convective and radiative fluxes, the temperature gradients of the medium and elements and finally, the typical velocity and dimensions of the radial and expansion/contraction motion of convective elements. In parallel we have also calculated the same quantities with the standard ML theory in which the ML parameter has been previously calibrated. All the results obtained with ML theory are recovered with the new theory but no scale parameters are adopted. We claim that the new theory is able to capture the essence of the convection in stellar interiors without a fine-tuned parameter inserted by hand.

The main achievement of the theory presented in this paper is not only to prove that satisfactory results can be achieved, as was already done by the ML theory, but more...
Results 3: Stellar models

Figure 9. The HRD of the 0.8, 1.0, 1.5, 2.0, and 2.5 $M_\odot$ stars with initial chemical composition $[X]=0.703$, $[Y]=0.280$, $[Z]=0.017$ calculated from the main sequence to advanced evolutionary stages using both the classical ML theory (the crossed lines) and the SFC theory (dotted lines of different colors). The 0.8, 1.0, 1.5 and 2.0 $M_\odot$ models are carried to a late stage of the RGB before core He-ignition (He-Flash), whereas the 2.5 $M_\odot$ is evolved up to very advanced stages of central He-burning ($Y_c \approx 0.1$). The stellar models are calculated with the Padova code and input physics used by Bertelli et al. (1994) and Bertelli et al. (2008), see also the text for more details. The models are meant to prove doubt that the SFC theory with no ML parameter is equivalent to the classical ML theory with calibrated ML parameter ($\Lambda_m = 1.68$ in our case).

Note: away from the well-calibrated cases, care should be exercised in which approach is the considered to be the reference.
Full stellar models: Overshooting

• The new theory does not yet include overshooting

• However, it derives the acceleration acquired by convective elements under the action of the buoyancy force in presence of the inertia of the displaced fluid and gravity.

• Therefore, it is also best suited to describe convective overshooting

• Extension of the atmospheric modelling to include overshooting is in preparation (Pasetto et al 2017)
Summary

• The correct treatment of convection is critical for stellar models throughout the H-R diagram
• The current standard approach using Mixing Length Theory requires
  – an additional relation not justified within the theory
  – with a calibration which is not universal
• A self consistent theory has been derived which allows the system of equations to be closed – this depends on
  – a formulation within co-moving coordinates
  – a new definition of the stability criterion
  – an identification of a growth-rate relation which allows the elimination of one of the variables in the formulation
• The new theory agrees closely with the MLT in the case of the Sun where the MLT is well-calibrated
• The new theory predicts sensible stellar evolutionary tracks, which may already be better than MLT outside where this is calibrated.
• The new theory can be extended to be applied broadly (geology, meteorology, oceanography) with the addition of viscosity terms