Two-dimensional time-dependent solution of Stix's equation for the distribution function of minority ions during ion cyclotron resonant heating

S. Dalla[a)]
JET Joint Undertaking, Abingdon, Oxfordshire, OX14 3EA, United Kingdom
and Centre for Computer and Mathematical Modelling, School of Electrical, Electronic and Information Engineering, South Bank University, London, SE1 0AA, United Kingdom

N. N. Ljepojevic[b)]
Centre for Computer and Mathematical Modelling, School of Electrical, Electronic and Information Engineering, South Bank University, London, SE1 0AA, United Kingdom

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The time-dependent flux-surface-averaged Fokker–Planck equation for the distribution function of minority ions during ion cyclotron resonant heating introduced by Stix in a classic paper [T. H. Stix, Nucl. Fusion **15**, 737 (1975)] is solved keeping two dimensions (2-D) in velocity space (speed and pitch-angle). An analysis of the applicability of the method of expansion of the distribution function f in Legendre polynomials of the pitch-angle, a method suggested by Stix and subsequently used by others, is given. A full numerical 2-D solution is also calculated. It is shown that the convergence of the Legendre polynomial expansion is very slow and non-uniform with respect to both particle energy and pitch-angle, making the method impractical when f is required at energies much higher than the background plasma thermal energy. However, the iterative sequences for the moments of f are found to converge very fast. In particular, a good approximation to the pitch-angle average of the distribution function is obtained already with two terms kept in the expansion, for a wide range of heating parameters. The validity of Stix's analytical one-dimensional approximations is analysed in detail. © *1997 American Institute of Physics.* [S1070-664X(97)02906-6]

I. INTRODUCTION

Ion cyclotron resonant heating (ICRH) is one of the schemes for plasma additional heating in tokamak experiments, and is used in the minority heating regime to accelerate particles of a low density ion species by means of radiofrequency (rf) waves at their cyclotron frequency or low order harmonics.[1,2] The collisional slowing-down of minority ions results in heating the background plasma.

During ICRH, the velocity distribution function of minority ions becomes strongly non-Maxwellian, a significant part of the ion energy being in the tail of the distribution. Therefore, a precise knowledge of the distribution function is needed for the determination of power transferred to the background plasma. Also, the non-Maxwellian character of the hot minority ions will influence the distribution functions of both the electrons and majority ions and thus the fusion reaction rate.

A classic work on the distribution function of minority ions during ICRH is the one by Stix.[1] He derived a Fokker–Planck equation for a homogeneous infinite magnetized plasma including, in addition to collisions, a so-called quasilinear term which describes the interaction of minority ions with the rf wave. His model aims at describing a small volume around a magnetic surface in a tokamak, in which plasma parameters are approximately homogeneous. In this picture, the distribution function f of minority ions is a function of speed v, the angle θ between the velocity and the constant direction of magnetic field (the so-called pitch-angle) and time t, i.e. $f=f(v,\theta,t)$. The function f is assumed to be independent of the third coordinate in velocity space, i.e. the phase of gyration along the magnetic field lines. The background plasma parameters, such as electron and majority ion density and temperature, and the intensity and direction of magnetic field, are assumed to remain constant during the heating.

Stix found analytical one-dimensional approximations to the steady-state solution of his equation, using two distinct methods. In the first of these, he introduced an expansion of the angular part of the distribution function in Legendre polynomials of even order of $\mu=\cos\theta$. He then kept only the first term in the expansion, corresponding to the assumption that the distribution function is not too far from being isotropic, and was able to derive a pitch-angle-independent analytical approximation to f. In the second method he introduced the coordinates $(v_\parallel, v_\perp)$, corresponding to velocity components in the direction of the magnetic field and in planes perpendicular to it, respectively. By assuming that for all particles $v_\perp \gg |v_\parallel|$, and integrating the whole equation over the variable $v_\parallel$, he obtained an equation for the averaged distribution $\bar{f}(v_\perp,t)=\int_{-\infty}^{+\infty} dv_\parallel\, f(v_\parallel,v_\perp,t)$ and solved it analytically.

After Stix's work, the so-called bounce-averaged theories were developed.[3,4] Here, a Fokker–Planck equation averaged over both gyrophase and the period of the test particles' orbit motion in the tokamak magnetic field is derived. This approach allows for an appropriate treatment of trapped particles, neglected in Stix's model. A number of codes which solve the bounce-averaged Fokker–Planck equation for the ICRH problem keeping the two dimensions (v,θ) in velocity space have been developed in recent years.[5,6]

[a)]Electronic mail: dallasc@sbu.ac.uk
[b)]Electronic mail: ljepojnn@sbu.ac.uk

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Stix's analytical solutions are still widely used in the analysis of ICRH experiments.[7–9] The Stix pitch-angle-independent approach has been generalized to include finite Larmor radius effects and second harmonic heating.[10,11]

Direct measurements of the distribution function have been made by means of Neutral Particle Analyzers (NPAs).[12,13] The NPA in the tokamak JET (Joint European Torus[14]) can detect minority ions with energies in the MeV region.[12] For comparison with the directly measured distribution functions, one-dimensional information, like the one provided by Stix's formulae, is not sufficient (typically NPAs measure f at a specified pitch-angle).

A systematic analysis of two-dimensional (2-D) time-dependent solutions of the Stix original equation is lacking in the literature and is the subject of the present paper. Such an analysis is needed in order to establish the meaning and the degree of approximation of the Stix one-dimensional solutions and other averaged approaches.[10] Also, an accurate 2-D time-dependent solution is required for the comparison with measurements from the NPA. In addition, as the bounce-averaged theories calculate 2-D distributions, a 2-D solution of the Stix model is needed to be able to compare quantitatively the two models.

The two-dimensional solution needed for the above purposes can be obtained by solving the Stix equation numerically using a 2-D grid in velocity space. In this paper, in addition to providing this purely numerical solution, we also apply the method of expansion into Legendre polynomials of the pitch-angle to the solution of the Stix equation. The use of the method comes as the natural extension of Stix's work and makes the comparison with the Stix pitch-angle-independent solution easier. Furthermore, the expansion into Legendre polynomials has the important advantage that the moments of the distribution function can be readily obtained, as they are related to the coefficients of the expansion by simple relations.

A few authors have worked towards obtaining a two-dimensional solution of the Stix equation. In Ref. 15 the first two terms of the Legendre polynomial expansion are kept in the steady-state solution, but the coefficients are calculated using a method appropriate for very low ICRH power densities only. An expansion in irreducible Hermite polynomials is used in Ref. 16 for the same problem. In Ref. 17 a larger number of terms in the Legendre polynomial expansion of the steady-state solution have been retained but convergence has been studied only for energies not too far from the thermal region. A numerical time-dependent two-dimensional solution of the Stix equation has been obtained in Ref. 18; however in that paper the emphasis is on the moments of the distribution and a detailed analysis of the distribution function itself is not given.

In this paper we apply the method of expansion in Legendre polynomials of the pitch-angle to the solution of Stix's equation, and demonstrate the limits of applicability of the method. We consider a wide energy range, which includes the highly non-Maxwellian tail of the distribution. Stix in his paper derived a set of equations for the first two coefficients in the expansion only. He recognized (p. 750 of Ref. 1) that the use of just two terms in the expansion is unsatisfactory when the heating powers are large and suggested that additional terms could be used. To be able to retain more terms, we derive an equation for the generic coefficient of the Legendre polynomial of order $2n$. We study the convergence of the solution by successively increasing the number of terms kept in the expansion.

As a further test, we calculate $f(v,\theta,t)$ by solving numerically the full Fokker–Planck equation in two dimensions in velocity space plus time, and compare it with the approximate solutions obtained using the Legendre polynomial expansion method. From this comparison we draw conclusions about the applicability of the method.

The layout of the paper is as follows. In Sec. II we describe the Stix model and in Sec. III we summarize Stix's one-dimensional results. The solution of the Stix equation by means of an expansion in Legendre polynomials of the pitch-angle in which an arbitrary number of terms are kept is described in Sec. IV. In Sec. V the convergence of partial sums is studied and a comparison with the full two-dimensional numerical solution of Stix's equation is given. In Sec. VI a comparison between the two-dimensional numerical solution and Stix's analytical expression for $\tilde{f}(v_\perp)$ is given. Conclusions are summarized in Sec. VII.

II. STIX'S MODEL FOR HEATING AT THE FUNDAMENTAL FREQUENCY IN THE SMALL LARMOR RADIUS LIMIT

In a homogeneous infinite plasma with a constant magnetic field, a minority species heated by an externally applied wave at the ion cyclotron frequency is described by its distribution function $f(v,\mu,t)$, where v is the particle speed, $\mu = \cos\theta$, θ being the angle between the velocity and the constant direction of the magnetic field (the so-called pitch-angle) and t is time. The distribution function is assumed to be independent of the phase of gyromotion.

The equation Stix derived for the evolution of f is[1]

$$\frac{\partial f}{\partial t}=C(f)+Q(f), \tag{1}$$

where:

$$C(f)=\frac{1}{v^2}\frac{\partial}{\partial v}\left[-\alpha v^2 f+\frac{1}{2}\frac{\partial}{\partial v}(\beta v^2 f)\right]+\frac{\gamma}{4v^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial f}{\partial \mu}\right],$$

$$Q(f)=\frac{3}{2}K\frac{1}{v^2}\left\{(1-\mu^2)\frac{\partial}{\partial v}\left[v\left(\frac{\partial}{\partial v}(vf)-\frac{\partial}{\partial \mu}(\mu f)\right)\right]-\frac{\partial}{\partial \mu}\left[\mu(1-\mu^2)\left(\frac{\partial}{\partial v}(vf)-\frac{\partial}{\partial \mu}(\mu f)\right)\right]\right\}.$$

Here $C(f)$ is the collisional term and $Q(f)$ is the so-called *quasi-linear* term, describing the changes in the distribution function due to particle acceleration by the rf wave.

The collisional diffusion coefficients $\alpha(v),\beta(v)$ and $\gamma(v)$ which appear in the above expression for $C(f)$ are the ones calculated by Chandrasekhar and Spitzer.[19,20] For a test minority particle colliding with particles of several background species (each denoted by a subscript j and of number density n_j, temperature T_j, mass m_j and charge number Z_j) they are given by

$$\alpha(v)=\langle\Delta v_{\parallel}\rangle(v)+\frac{1}{2v}\gamma(v), \tag{2}$$

$$\beta(v)=\sum_j \frac{1}{v}[\delta_j G(x_j)], \tag{3}$$

$$\gamma(v)=\sum_j \frac{1}{v}\delta_j[\Phi(x_j)-G(x_j)], \tag{4}$$

$$\langle\Delta v_{\parallel}\rangle(v)=-\sum_j \frac{\delta_j}{v_{tj}^2}\left(1+\frac{m_m}{m_j}\right)G(x_j), \tag{5}$$

where $x_j=v/v_{tj}$ with $v_{tj}=(2T_j/m_j)^{1/2}$, the thermal velocity of background particles of the species j. The functions $\Phi(x)$ and $G(x)$ are

$$\Phi(x)=\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy,$$
$$G(x)=\frac{\Phi(x)-x\,(d\Phi/dx)}{2x^2}, \tag{6}$$

and the coefficients δ_j are

$$\delta_j=\frac{8\pi n_j Z_m^2 Z_j^2 e^4 \ln\Lambda_j}{m_m^2},$$

where $\ln\Lambda_j$ is the Coulomb logarithm:

$$\ln\Lambda_j=\ln\frac{\lambda_D}{b_{0j}},\quad \lambda_D=\left(\sum_j \frac{4\pi n_j e^2}{T_j}\right)^{-1/2},$$

$$b_{0j}=\frac{Z_m Z_j e^2}{\mu_j \overline{g_j^2}}.$$

In the above formulae m_m is the mass of minority ions, Z_m their charge, λ_D the Debye length, $\mu_j=m_m m_j/(m_m+m_j)$ the reduced mass and $\overline{g_j^2}$ the mean square value of the relative velocity between a minority ion and a particle of the background species j.

The collisional term described above takes into account collisions between a minority particle and particles of the background plasma species. The latter are assumed to have Maxwellian distribution functions. Collisions between minority ions are neglected, due to the low density of the minority species with respect to the other background species. The temperature of background species is assumed to remain constant in time. This is clearly a limit of this description, as background plasma species get heated by collisions with minority ions. In the following we assume the background plasma to consist of one majority ionic species and electrons.

The expression for $Q(f)$ given in Eq. (1) has been obtained by Stix by averaging the space-localized quasi-linear operator derived in Ref. 21 over the volume around a magnetic surface. Also, the approximation of the small Larmor radius is introduced and heating at the fundamental frequency only is considered. The constant K in the quasi-linear term in Eq. (1) is related to $\langle P\rangle$, the average power per unit volume absorbed by minority particles from the rf wave in the volume under consideration, according to the relation

$$K=\frac{\langle P\rangle}{3m_m n_m}, \tag{7}$$

n_m being the number density of minority ions.

III. STIX'S ANALYTICAL ONE-DIMENSIONAL SOLUTIONS

By introducing approximations which reduce the dependence of the distribution function to only one velocity space variable, Stix was able to find analytical expressions for the steady state solution of Eq. (1). This was achieved in the (v,μ) coordinate system by neglecting the pitch-angle dependence of the distribution function, and in the $(v_{\parallel},v_{\perp})$ system by integrating the distribution function over the variable $v_{\parallel}$. We now summarize his results.

A. Pitch-angle-independent solution

If the angular dependence of the distribution function in velocity space is neglected, the steady-state solution of Eq. (1) can be found immediately by imposing $\partial f/\partial t=0$, $\partial f/\partial\mu=0$ and analytically integrating the equation. The following boundary conditions are used:

$$\lim_{v\to\infty} f(v)=\lim_{v\to\infty}\frac{df}{dv}(v)=0,$$

leading to the following expression for $f(v)$:[1]

$$\ln\frac{f(v)}{f_0}=-\int_0^v dv'\frac{-\alpha v'^2+\frac{1}{2}(d/dv')(\beta v'^2)}{\left(\frac{1}{2}\beta+K\right)v'^2}, \tag{8}$$

where f_0 is a constant which is determined by imposing a normalization condition on f. The integral in this equation has been carried out explicitly by Stix by introducing approximating forms for the functions Φ and G which enter the definitions of the diffusion coefficients α and β. The approximations used are as follows:

$$G(x_e)=\epsilon x_e,\quad \Phi(x_e)=\epsilon x_e(3+2x_e^2), \tag{9}$$

$$G(x_i)=\frac{\epsilon x_i}{1+2\epsilon x_i^3},\quad \Phi(x_i)=\frac{\epsilon x_i(3+2x_i^2)}{1+2\epsilon x_i^3}, \tag{10}$$

where the subscript i refers to majority ions and e to electrons, and $\epsilon=2/(3\pi^{1/2})$. The use of the above expressions has allowed Stix to find an explicit formula for $f(v)$ [Eq. (34) of Ref. 1] and the following formula for the tail temperature of the distribution function:

$$T_{tail}=T_e(1+\xi),\quad \xi=\frac{m_m\langle P\rangle}{8\pi^{1/2}n_e n_m Z_m^2 e^4 \ln\Lambda}v_{te}, \tag{11}$$

where the tail temperature has been defined as the asymptotic value at large energies of the effective temperature $T_{\mathrm{eff}}(E)=-[(d/dE)(\ln f(v))]^{-1}$ (here E is particle kinetic energy).

The validity of the Stix pitch-angle independent solution as a good representation of the distribution function is limited to very low ICRH power cases, where f is not too far from being isotropic in velocity space. However, his solution for $f(v)$ and the asymptotic expression for the tail temperature given by Eq. (11) are frequently used as approximations to the pitch-angle average of f even in high power cases in which the distribution is anisotropic. This point will be further discussed in Sec. V.

B. $\tilde{f}(v_\perp)$ approximation

Stix re-wrote Eq. (1) in the $(v_\parallel, v_\perp)$ coordinates, where $v_\parallel$ and $v_\perp$ are the components of velocity in the direction of the magnetic field and in planes perpendicular to it, respectively. He then assumed that for all minority ions $v_\perp \gg |v_\parallel|$, i.e. $v \approx v_\perp$, and introduced the following integral of the distribution function:

$$\tilde{f}(v_\perp, t) = \int_{-\infty}^{+\infty} dv_\parallel \, f(v_\parallel, v_\perp, t).$$

By integrating his equation over the variable $v_\parallel$ he then found an equation for $\tilde{f}(v_\perp)$, yielding the following steady-state solution:

$$\ln \frac{\tilde{f}(v_\perp)}{\tilde{f}_0} = -\int_0^{v_\perp} dv'_\perp \frac{-\alpha v'_\perp + \frac{1}{2}\frac{d}{dv'_\perp}(\beta v'_\perp) + \frac{\gamma}{4}}{\left(\frac{1}{2}\beta + \frac{3}{2}K\right) v'_\perp}. \tag{12}$$

Again introducing the approximating forms for the functions Φ and G given in Eqs. (9)–(10), Stix was able to further simplify this expression, obtaining an explicit form for $\tilde{f}(v_\perp)$ [Eq. (38) of Ref. 1] and the following asymptotic expression for its temperature:

$$T_{tail}^{\perp} = -\left[\frac{d}{dE}(\ln \tilde{f}(v_\perp))\right]^{-1} = T_e\left(1 + \frac{3}{2}\xi\right), \tag{13}$$

where the parameter ξ has been defined in Eq. (11) above.

IV. SOLUTION OF STIX'S EQUATION USING THE METHOD OF EXPANSION IN LEGENDRE POLYNOMIALS OF THE PITCH-ANGLE

In this section we introduce an expansion of the distribution function into the Legendre polynomials of μ, in order to facilitate the solution of Eq. (1). This approach was suggested by Stix in his paper, where he at first retained the first two terms in the expansion and derived the set of equations for the corresponding coefficients. In the actual solution for the distribution function in the velocity range above the background ion thermal velocity, he then kept only the first term of the expansion and obtained his pitch-angle-independent solution as given by Eq. (8) above.

Here we generalize Stix's approach in that we derive the evolution equation for the generic expansion coefficient and thus are able to retain an arbitrary number of terms in the expansion. We write the distribution function as

$$f(v, \mu, t) = \sum_{n=0}^{+\infty} A_{2n}(v, t) P_{2n}(\mu), \tag{14}$$

where only the even Legendre polynomials are considered due to the symmetry of f in θ around $\theta = \pi/2$. Substitution of this expansion into Eq. (1), multiplication by Legendre polynomials in succession and integration over $d\mu$ yields an infinite system of coupled differential equations for the expansion coefficients. We find that the equation of evolution for the generic coefficient $A_{2n}(v,t)$ of the expansion in Legendre polynomials is given by

$$\begin{aligned}\frac{\partial A_{2n}}{\partial t} &= \left[\frac{1}{2}\beta + a_n K\right]\frac{\partial^2 A_{2n}}{\partial v^2} + \left[c_1(v) + 2a_n \frac{K}{v}\right]\frac{\partial A_{2n}}{\partial v} + \left[c_0(v) - 2n(2n+1)a_n \frac{K}{v^2} - \frac{n}{2}(2n+1)\frac{\gamma}{v^2}\right]A_{2n} \\ &\quad - b_n K \frac{\partial^2 A_{2n-2}}{\partial v^2} + (4n-3)b_n \frac{K}{v}\frac{\partial A_{2n-2}}{\partial v} - 4n(n-1)b_n \frac{K}{v^2}A_{2n-2} - d_n K \frac{\partial^2 A_{2n+2}}{\partial v^2} \\ &\quad - (4n+5)d_n \frac{K}{v}\frac{\partial A_{2n+2}}{\partial v} - (2n+3)(2n+1)d_n \frac{K}{v^2}A_{2n+2},\end{aligned} \tag{15}$$

where

$$a_n = \frac{3}{4}\left[1 - \frac{1}{(4n+3)(4n-1)}\right],$$

$$b_n = \frac{3n(2n-1)}{(4n-3)(4n-1)}, \quad d_n = \frac{3(n+1)(2n+1)}{(4n+5)(4n+3)},$$

$$c_0(v) = \frac{1}{v^2}\left[\frac{d}{dv}(-\alpha v^2) + \frac{1}{2}\frac{d^2}{dv^2}(\beta v^2)\right],$$

$$c_1(v) = \frac{1}{v^2}\left[-\alpha v^2 + \frac{d}{dv}(\beta v^2)\right].$$

We see that the system of differential equations generated by Eq. (15) is of a tridiagonal form, in the sense that the coefficient A_{2n} is directly coupled with A_{2n-2} and A_{2n+2} only. For $n=0$ this simplifies further to A_0 being directly coupled with A_2 only. It is also interesting to notice that for $K = 0$ the system fully decouples.

In practice, the expansion given by Eq. (14) is truncated to n_t terms. The finite set of n_t equations for the corresponding coefficients of the expansion is obtained by substituting into Eq. (15) the values $n=0, \ldots, n_t-1$ and by imposing $A_{2n_t}\equiv 0$ in the equation for A_{2n_t-2}. For the truncated series to represent accurately the solution of the original equation, the number of retained terms needs to be chosen adequately. This requires an analysis of the convergence of the sequence of distribution functions obtained with $n_t = 1, 2, 3, \ldots$. This analysis is presented in Sec. V below.

Here we explicitly derive the system of equations for the coefficients for the case $n_t = 5$, which is the largest value of n_t considered in this paper. By setting $n=0$, 1, 2, 3, 4 into Eq. (15) and imposing $A_{10}\equiv 0$ in the equation for A_8, we find the following set of equations for the coefficients A_0, A_2, A_4, A_6, A_8:

$$\frac{\partial A_0}{\partial t}=\left[\frac{1}{2}\beta+K\right]\frac{\partial^2 A_0}{\partial v^2}+\left[c_1(v)+2\frac{K}{v}\right]\frac{\partial A_0}{\partial v}+c_0(v)A_0 -\frac{1}{5}K\frac{\partial^2 A_2}{\partial v^2}-\frac{K}{v}\frac{\partial A_2}{\partial v}-\frac{3}{5}\frac{K}{v^2}A_2, \quad (16a)$$

$$\frac{\partial A_2}{\partial t}=\left[\frac{1}{2}\beta+\frac{5}{7}K\right]\frac{\partial^2 A_2}{\partial v^2}+\left[c_1(v)+\frac{10}{7}\frac{K}{v}\right]\frac{\partial A_2}{\partial v}+\left[c_0(v) -\frac{30}{7}\frac{K}{v^2}-\frac{3}{2}\frac{\gamma}{v^2}\right]A_2-K\frac{\partial^2 A_0}{\partial v^2}+\frac{K}{v}\frac{\partial A_0}{\partial v} -\frac{2}{7}K\frac{\partial^2 A_4}{\partial v^2}-\frac{18}{7}\frac{K}{v}\frac{\partial A_4}{\partial v}-\frac{30}{7}\frac{K}{v^2}A_4, \quad (16b)$$

$$\frac{\partial A_4}{\partial t}=\left[\frac{1}{2}\beta+\frac{57}{77}K\right]\frac{\partial^2 A_4}{\partial v^2}+\left[c_1(v)+\frac{114}{77}\frac{K}{v}\right]\frac{\partial A_4}{\partial v} +\left[c_0(v)-\frac{1140}{77}\frac{K}{v^2}-5\frac{\gamma}{v^2}\right]A_4-\frac{18}{35}K\frac{\partial^2 A_2}{\partial v^2} +\frac{18}{7}\frac{K}{v}\frac{\partial A_2}{\partial v}-\frac{144}{35}\frac{K}{v^2}A_2-\frac{45}{143}K\frac{\partial^2 A_6}{\partial v^2} -\frac{45}{11}\frac{K}{v}\frac{\partial A_6}{\partial v}-\frac{1575}{143}\frac{K}{v^2}A_6, \quad (16c)$$

$$\frac{\partial A_6}{\partial t}=\left[\frac{1}{2}\beta+\frac{41}{55}K\right]\frac{\partial^2 A_6}{\partial v^2}+\left[c_1(v)+\frac{82}{55}\frac{K}{v}\right]\frac{\partial A_6}{\partial v}+\left[c_0(v) -\frac{1722}{55}\frac{K}{v^2}-\frac{21}{2}\frac{\gamma}{v^2}\right]A_6-\frac{5}{11}K\frac{\partial^2 A_4}{\partial v^2}+\frac{45}{11}\frac{K}{v}\frac{\partial A_4}{\partial v} -\frac{120}{11}\frac{K}{v^2}A_4-\frac{28}{85}K\frac{\partial^2 A_8}{\partial v^2}-\frac{28}{5}\frac{K}{v}\frac{\partial A_8}{\partial v} -\frac{1764}{85}\frac{K}{v^2}A_8, \quad (16d)$$

$$\frac{\partial A_8}{\partial t}=\left[\frac{1}{2}\beta+\frac{71}{95}K\right]\frac{\partial^2 A_8}{\partial v^2}+\left[c_1(v)+\frac{142}{95}\frac{K}{v}\right]\frac{\partial A_8}{\partial v} +\left[c_0(v)-\frac{5112}{95}\frac{K}{v^2}-18\frac{\gamma}{v^2}\right]A_8-\frac{28}{65}K\frac{\partial^2 A_6}{\partial v^2} +\frac{28}{5}\frac{K}{v}\frac{\partial A_6}{\partial v}-\frac{1344}{65}\frac{K}{v^2}A_6. \quad (16e)$$

Systems for cases of smaller n_t are easily obtained from this one by setting to zero the coefficient A_{2n_t} and higher. In particular we notice that the system of Eqs. (16a)–(16e) with $A_4 = A_6 = A_8\equiv 0$ (i.e., $n_t = 2$) is identical to Eqs. (24)–(25) of Ref. 1, where the notation A and B is used instead of A_0 and A_2.

We solve the truncated system of equations for the coefficients numerically using an explicit finite difference method; the full non-approximate expressions for the functions G and Φ as by Eq. (6) are used in the calculation of the collisional diffusion coefficients. The boundary conditions are [see Eqs. (20)–(23)]:

$$\frac{\partial A_{2n}}{\partial v}\Big|_{v=0}=0, \quad \forall t, \quad n=0,\ldots,4,$$

$$A_{2n}|_{v=0}=0, \quad \forall t, \quad n=1,\ldots,4,$$

$$\lim_{v\to\infty} A_{2n}(v,t)=0, \quad \forall t, \quad n=0,\ldots,4.$$

At the initial time the distribution function of minority ions is assumed to be a Maxwellian with temperature equal to the background ion temperature, which results in the initial conditions

$$A_0(v,t=0)=\exp\left(-\frac{v^2}{v_{tm}^2}\right),$$

$$A_{2n}(v,t=0)\equiv 0 \quad n=1,\ldots,4,$$

where v_{tm} is the thermal velocity of minority ions at $t=0$.

The integration in the time domain is carried out until the steady-state is reached. Typically the steady-state is obtained in a time $t=3\tau_{sd}$, where τ_{sd} is the Spitzer slowing-down time for a particle of speed in the range $v_{ti}\ll v\ll v_{te}$, given by[20]

$$\tau_{sd}=\frac{3\pi^{1/2}v_{te}^3}{2(1+m_m/m_e)\delta_e}. \quad (17)$$

For $n_t=1$, i.e., when the system of Eqs. (16a)–(16e) is solved imposing $A_2=A_4=A_6=A_8\equiv 0$, the Stix steady-state pitch-angle-independent solution [given by Eq. (8) above] is recovered. We have used this as a test for our numerical procedure.

V. CONVERGENCE OF LEGENDRE POLYNOMIAL EXPANSION AND COMPARISON WITH FULL SOLUTION ON A 2-D GRID

As a way to assess its convergence, we truncate the Legendre polynomial expansion in succession, i.e. we consider the cases $n_t=1$, 2, 3, ... , where n_t is the number of retained terms. For each n_t we solve the corresponding set of equations for the unknowns $A_0, A_2, \ldots, A_{2n_t-2}$. We see that at each stage one more coefficient, A_{2n_t-2}, which was set to zero at the previous stage, gets involved and its first approximation is calculated; the coefficients obtained at the previous stage, i.e. $A_0, A_2, \ldots, A_{2n_t-4}$, are updated. We denote the coefficients calculated at the stage n_t by $A_0^{(n_t)}, A_2^{(n_t)}, \ldots, A_{2n_t-2}^{(n_t)}$. These are used to calculate the n_t-th approximation to the distribution function, i.e.,

$$f^{(n_t)}(v,\mu,t)=\sum_{k=0}^{n_t-1} A_{2k}^{(n_t)}(v,t)P_{2k}(\mu). \tag{18}$$

At each stage the approximation for f gets updated through the updating of the coefficients $A_0, A_2, \ldots, A_{2n_t-4}$ and, more importantly in our case, by adding a new term $A_{2n_t-2}^{(n_t)} P_{2n_t-2}$ in the sum (18). We study the convergence of the sequence $\{f^{(n_t)}, n_t = 1, 2, 3, \ldots\}$. We expect that the limit of $f^{(n_t)}$ as $n_t \to \infty$ will give the distribution function f.

As a further test of convergence of the Legendre expansion, we also calculate $f(v,\mu,t)$ by solving numerically the original equation [Eq. (1)] on a two-dimensional grid. If derivatives in Eq. (1) are carried out explicitly and the variables (v,θ) are introduced, where θ is the pitch-angle, one obtains

$$\begin{aligned}\frac{\partial f}{\partial t} &= \left[\frac{1}{2}\beta+\frac{3}{2}K\sin^2\theta\right]\frac{\partial^2 f}{\partial v^2}+\left[c_1(v)\right.\\ &\left.+\frac{3}{2}K\frac{1+\cos^2\theta}{v}\right]\frac{\partial f}{\partial v}+\frac{1}{v^2}\left[\frac{3}{2}K\cos^2\theta+\frac{\gamma}{4}\right]\frac{\partial^2 f}{\partial \theta^2}\\ &+3K\frac{\sin\theta\cos\theta}{v}\frac{\partial^2 f}{\partial v\partial\theta}-\frac{1}{v^2}\frac{\cos\theta}{\sin\theta}\\ &\times\left[\frac{3}{2}K(1-2\cos^2\theta)-\frac{\gamma}{4}\right]\frac{\partial f}{\partial\theta}+c_0(v)f.\end{aligned} \tag{19}$$

We have solved this equation by means of an explicit finite difference method, using a two-dimensional grid in velocity space. Because of the symmetry of the distribution function in θ around $\theta=\pi/2$, the equation needs to be solved on the interval $\theta \in [0,\pi/2]$ only. The boundary conditions are as follows:

$$\left.\frac{\partial f}{\partial v}\right|_{v=0}=0, \quad \forall\theta, \quad \forall t, \tag{20}$$

$$\left.\frac{\partial f}{\partial \theta}\right|_{v=0}=0, \quad \forall\theta, \quad \forall t, \tag{21}$$

$$\lim_{v\to\infty} f(v,\theta,t)=0, \quad \forall\theta, \quad \forall t, \tag{22}$$

$$\left.\frac{\partial f}{\partial \theta}\right|_{\theta=0}=\left.\frac{\partial f}{\partial \theta}\right|_{\theta=\pi/2}=0, \quad \forall v, \quad \forall t, \tag{23}$$

and for the initial condition we have taken a Maxwellian, as in Sec. IV.

We consider the heating of a hydrogen minority in deuterium, for values of the power per unit volume deposited by ICRH, $\langle P\rangle$, between 0.1 and 1.0 MW/m^3. The electron density is $n_e = 2.65\times 10^{19}$ m^{-3}, the ratio of minority ion to electron density $n_H/n_e = 0.06$ and the background temperatures $T_e=T_i=4$ keV.

We now study the convergence of (a) the pitch-angle average of f; (b) $f(v,\theta)$ at the values of pitch-angle $\theta=\pi/2, \pi/4$ and 0.

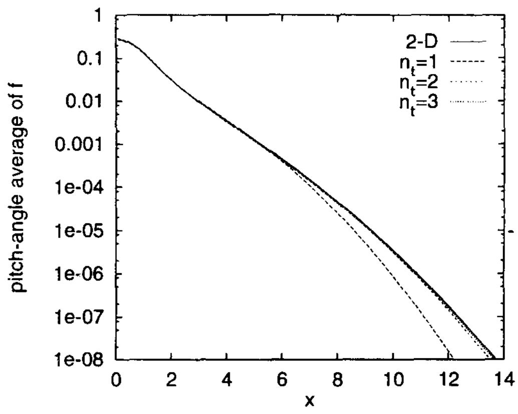

FIG. 1. Pitch-angle average of the steady-state f, versus $x=v/v_{tm}$. The heating and plasma parameters are $\langle P\rangle$ = 0.1 MW/m^3, n_e = 2.65$\times$10^{19} m^{-3}, $T_e = T_i = T_m(t=0)$ = 4 keV; hydrogen minority in deuterium with n_H/n_e=0.06.

A. Pitch-angle average of *f*

The *pitch-angle average* of $f(v,\mu,t)$ is defined as

$$\langle f(v,\mu,t)\rangle=\frac{1}{2}\int_{-1}^{+1} d\mu\, f(v,\mu,t).$$

When f is expanded in terms of Legendre polynomials of the pitch-angle [i.e., f is given by Eq. (14)], using the orthogonality property of Legendre polynomials one finds

$$\langle f(v,\mu,t)\rangle=A_0(v,t). \tag{24}$$

Therefore the first term in the expansion, $A_0(v,t)$, represents the pitch-angle average of the distribution function.

In Fig. 1 the steady-state A_0 for the cases n_t = 1, 2 and 3 as calculated from the corresponding systems of equations, for $\langle P\rangle$=0.1 MW/m^3, are plotted. Here a dimensionless variable $x=v/v_{tm}$ is introduced, where v_{tm} is the minority ion thermal velocity before the heating. In the same figure the pitch-angle average of the steady-state distribution function obtained from the solution of Eq. (19) on a two-dimensional grid is shown (solid line). The Stix explicit formula [Eq. (34) of Ref. 1] gives values coinciding with the n_t = 1 curve and therefore is not plotted in Fig. 1.

It can be seen that the addition of terms beyond the first one in the expansion modifies A_0 considerably. When A_2 only is added in the calculation (i.e., n_t=2), A_0 has a temperature in the high energy range which is higher than the one derived in the case $A_2\equiv 0$ (i.e., n_t=1). The addition of a further term, A_4, (i.e., n_t=3) in the expansion does not modify A_0 appreciably, and the $A_0(n_t=3)$ curve is very close to the pitch-angle average obtained from the 2-D solution. The same features are found in high ICRH power density cases, as is shown in Fig. 2 for $\langle P\rangle$=1.0 MW/m^3. Therefore we conclude that a satisfactory approximation to the pitch-angle-average of the distribution function (which determines the angle-independent moments of f, such as the total energy of the minority component, and the fusion reactivity) can be obtained by using a Legendre polynomial expansion in which two terms are retained, whereas one term only is definitely insufficient.

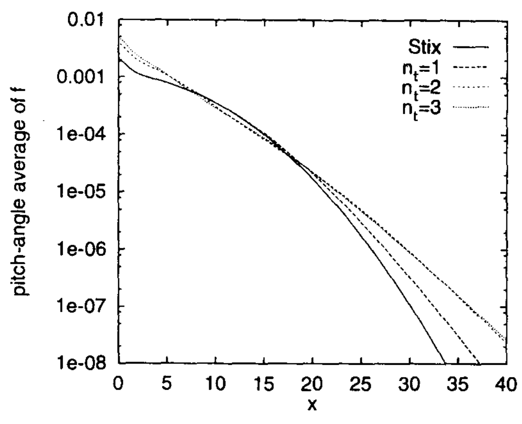

FIG. 2. Pitch-angle average of the steady-state f, versus $x=v/v_{tm}$, for power density $\langle P\rangle = 1.0\ \mathrm{MW/m^3}$. The other parameters are as in Fig. 1. The solid line is from Stix's explicit formula [Eq. (34) of Ref. 1].

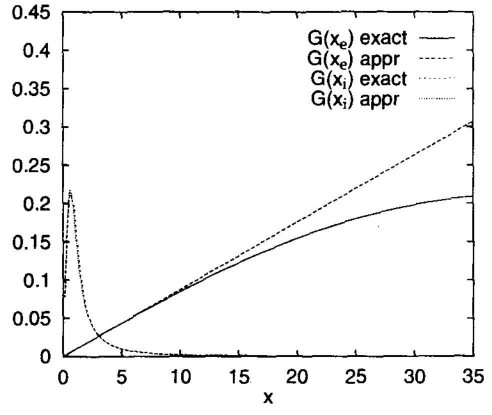

FIG. 3. Comparison between the exact and approximate forms for the function G, as given by Eqs. (6) and (9)–(10), respectively.

Our results contradict claims that a good approximation to the pitch-angle average of the distribution function can be obtained by keeping only the first term in a Legendre polynomial expansion also in cases in which the first term only is not sufficient to describe the distribution function itself.[10,11] We also note that in the approach of Ref. 15 two terms were kept in the expansion for f, but A_0 was calculated by setting $A_2 \equiv 0$ and was not subsequently modified. Therefore the pitch-angle-average of this two term approach is the same as Stix's.

In Table I the values of the tail temperature of A_0 in the high energy region are given, for the cases corresponding to keeping only the first term in the expansion ($n_t = 1$) and keeping two terms ($n_t = 2$). We see from the table that the tail temperatures for the latter case are about 30% bigger than for the former one. The values of the Stix parameter ξ and of the tail temperatures according to the Stix analytical expression $T_{tail}=T_e(1+\xi)$ [Eq. (11) above] are also given in the table.

We notice that as the ICRH power increases, a discrepancy between the value of T_{tail} from the Stix analytical formula and that obtained from the calculation for $n_t = 1$ appears, giving a total discrepancy between the $n_t = 2$ and the Stix analytical tail temperature of 50%. This feature is also noticeable from Fig. 2, which shows a discrepancy between the $n_t=1$ and the Stix explicit formula curve. This is a consequence of Stix's use of the approximating forms for the functions $G(x_e)$ and $\Phi(x_e)$ as given by Eq. (9), in the derivation of his explicit formulae for A_0 and the tail temperature. The approximate form for $G(x_e)$ is accurate only in the limit $v \ll v_{te}$, as is shown in Fig. 3. A similar situation takes place for $\Phi(x_e)$, whereas $G(x_i)$ and $\Phi(x_i)$ are quite adequately represented by Eq. (10). At high ICRH powers a considerable number of minority ions reaches velocities close to v_{te} and the use of the approximate forms therefore introduces an additional error in the calculation.

TABLE I. Temperature of pitch-angle average of f at various ICRH powers.

$\langle P\rangle$ (MW/m^3)	ξ	$T_e(1+\xi)$ (keV)	T_{tail} (keV), n_t=1	T_{tail} (keV), n_t=2
0.1	9.8	43	45	60
0.5	49.3	201	225	300
1.0	98.5	398	483	622

B. $f(v,\theta)$ at the values of pitch-angle $\theta=\pi/2, \pi/4$ and 0

In Fig. 4 a plot of the coefficients of the expansion for the case n_t=5 is given for the steady-state. A_2 and A_6 are negative for all values of x. We notice that at low energies the coefficients decrease very fast, whereas at large values of energy all five coefficients are of the same order of magnitude. This indicates that at high energies the order of magnitude of neglected terms of the expansion may be the same as that of the retained terms, i.e. that the Legendre expansion may have been truncated too early. Figure 4 clearly shows that as the energy considered increases, a larger number of terms in the expansion need to be retained, i.e., the convergence is non-uniform.

This fact is evident also from Fig. 5(a-c), in which plots of the total distribution, obtained by substituting the coefficients calculated in succession for n_t=1, 2, 3, 4 and 5 into

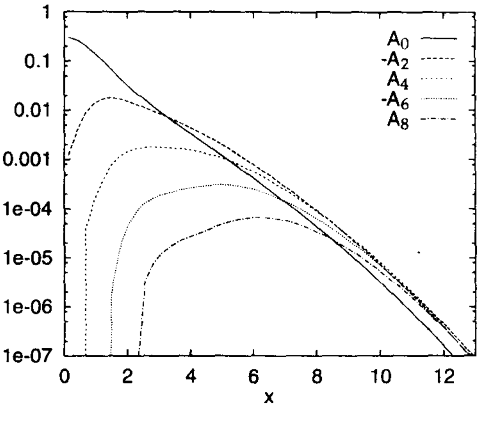

FIG. 4. Coefficients of Legendre polynomial expansion for $n_t = 5$. Plasma and heating parameters are as in Fig. 1.

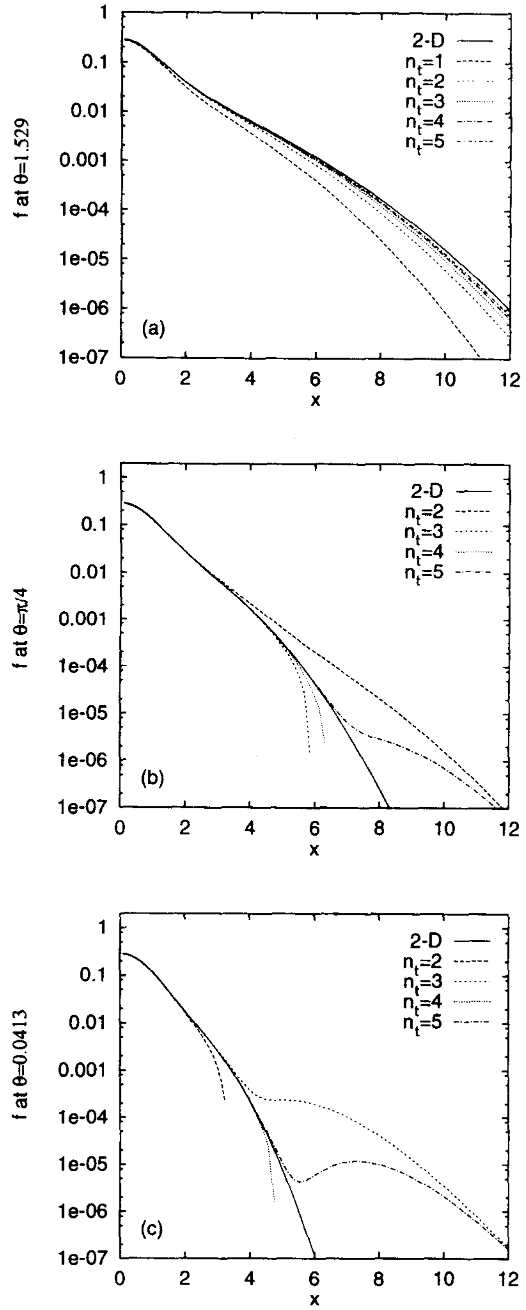

FIG. 5. $f(x,\theta)$ in steady-state for $\theta=\pi/2$ (a), $\theta=\pi/4$ (b) and $\theta=0$ (c). Plasma and heating parameters are as in Fig. 1.

Eq. (18), are given for pitch-angles $\theta=\pi/2$, $\pi/4$ and 0. The distribution function obtained by solving Eq. (19) on a two-dimensional grid is also plotted (solid line). As one can see, the convergence is fast only in the low energy range. In the figures corresponding to $\theta=\pi/4$ and 0, some plots of the distribution function from the Legendre polynomial expansion are terminated at low values of energy because f starts assuming negative values. This is a clear indication of the inadequacy in the high energy range of an expansion in which only the first few terms are kept.

The value of ICRH power density of $\langle P\rangle$ = 0.1 MW/m^3, considered in Figs. 4–5, is "low" for JET standards. Convergence of the Legendre polynomial expansion for higher power densities worsens and requires a larger number of terms even for energies not much larger than the thermal ones. In Ref. 17 the method of expansion in Legendre polynomials has been used to derive a steady-state solution to Stix's equation with a finite Larmor radius. In that work it was claimed that the first 11 terms in the expansion would give a good enough approximation to f. However, the convergence of the distribution function expansion after 11 terms was proven only at a single energy, corresponding to our $x=v/v_{tm}=4$ (Fig. 5 of Ref. 17), which is in a rather low energy range. As can be seen from our Fig. 2 and Fig. 7, at high heating power densities the range of values of x of interest extends far beyond $x=4$. Our conclusion is that the applicability of the Legendre polynomial expansion to deriving the full two-dimensional distribution function is limited to the energies not too far from thermal (for all pitch-angles), with a possibility of going to higher energies for pitch-angles in the vicinity of $\theta=\pi/2$, where the anisotropy is less strong.

The failure of the Legendre polynomial expansion to represent f at all velocities is due to the very strong anisotropy at large energies. Physically, this results from particles being heated in a preferential direction (the perpendicular direction) and collisions becoming less frequent and therefore less effective in isotropizing the distribution, as the energy increases. Anisotropy of the distribution at high energy is important in applications such as wave–particle interaction[17] and hot plasma stability.[22] For these applications and for the comparison with NPA measurements a full numerical solution on a two-dimensional grid is needed.

It is interesting to notice that while the distribution function itself is not well represented by the Legendre polynomial expansion, a good approximation to its pitch-angle average A_0 is obtained from an expansion of rather low dimension ($n_t=2$). In other words, while the convergence of the sequence $\{f^{(n_t)},\ n_t=1,2,\ldots\}$ is slow and nonuniform, the sequence $\{A_0^{(n_t)},\ n_t=1,2,\ldots\}$ converges very fast. The latter fortunate property is a consequence of the tridiagonal structure of the system for the coefficients and applies to other coefficients beyond A_0. In Fig. 6 we show the convergence of the coefficient A_2, which together with A_0 determines the pressure tensor.

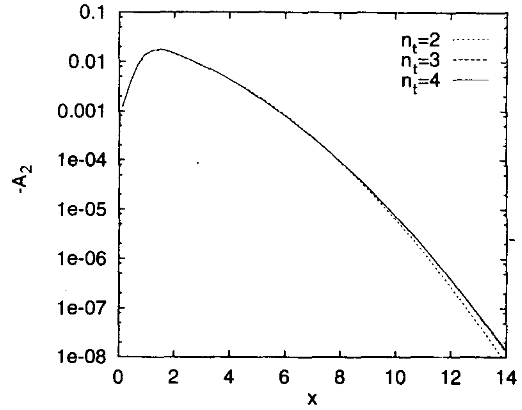

FIG. 6. The $-A_2$ in steady-state for the same parameters as in Fig. 1.

Many common applications such as the calculation of the energy content, fusion and other reaction rates, current, pressure tensor, etc., involve *integrals* of the distribution function convoluted with physical operators which are simple functions of the pitch-angle. In these applications the quantities involved can be expressed in terms of the first few coefficients of the Legendre expansion, and we conclude that in these cases the Legendre polynomial expansion is very useful.

We note that there is no contradiction in the different convergence rates of the sequences $\{A_{2k}^{(n_t)}, n_t=k+1, k+2, \ldots\}$ and the sequence $\{f^{(n_t)}, n_t=1, 2, \ldots\}$. The fast convergence of the former does not imply a fast convergence of the latter, in fact it does not guarantee that the latter sequence converges at all. For example, in the case of a system for the coefficients of diagonal structure, in which the equations are uncoupled, the coefficients are immediately determined accurately, but this is no guarantee of convergence of the series (14), i.e. of the sequence $\{f^{(n_t)}, n_t=1, 2, \ldots\}$. Mathematically speaking, the convergence of the sequences $\{A_{2k}^{(n_t)}, n_t=k+1, k+2, \ldots\}$ is a necessary but not sufficient condition for the convergence of $\{f^{(n_t)}, n_t=1, 2, \ldots\}$.

VI. COMPARISON BETWEEN THE TWO-DIMENSIONAL SOLUTION AND STIX'S $\tilde{f}(v_\perp)$

We now examine the second of the Stix analytical approximations, $\tilde{f}(v_\perp)$, which has been derived by assuming that for all minority ions $v_\perp \gg |v_\parallel|$, i.e., $v \approx v_\perp$. We first calculate $\tilde{f}(v_\perp)$ as given by Eq. (12), using the exact forms for the functions G and Φ [Eq. (6)] which appear in the definitions of the collisional diffusion coefficients. We also consider $\tilde{f}(v_\perp)$ from the Stix explicit formula [Eq. (38) of Ref. 1], which he obtained by introducing the approximate forms for G and Φ given by Eqs. (9)–(10), and which we will denote in the following as $\tilde{f}(v_\perp)_{appr}$. This explicit formula is frequently used in the analysis of ICRH experiments for cases of very high ICRH power density. We then compare $\tilde{f}(v_\perp)$ and $\tilde{f}(v_\perp)_{appr}$ with the steady-state distribution function obtained from the solution of the equation on a 2-D grid, at pitch-angle $\theta=\pi/2$.

We find that at low ICRH powers (e.g., $\langle P\rangle=0.1$ MW/m^3) $\tilde{f}(v_\perp)$ has the same slope in the tail as the 2-D solution at pitch-angle $\theta=\pi/2$, but a disagreement between the two is found in the thermal region, as expected. For these ICRH power densities $\tilde{f}(v_\perp)$ and $\tilde{f}(v_\perp)_{appr}$ coincide.

In Fig. 7 the comparison is presented for a high ICRH power case ($\langle P\rangle = 1.0$ MW/m^3). The solid line corresponds to the 2-D solution at $\theta=\pi/2$ and the dashed line, almost coincident with it, to $\tilde{f}(v_\perp)$ from Eq. (12). The dotted line in the plot corresponds to the Stix explicit formula for $\tilde{f}(v_\perp)_{appr}$. The tail temperature of the latter distribution, as calculated for $x \geq 30$, is T_{tail} = 591 keV, in agreement with the Stix formula $T_{tail}^{\perp}=T_e(1+\frac{3}{2}\xi)$. The tail temperature of the 2-D solution [and of $\tilde{f}(v_\perp)$] is instead T_{tail} = 860 keV. The discrepancy between the two is again a consequence of the use of the approximate forms for G and Φ in a region in which $v \approx v_{te}$.

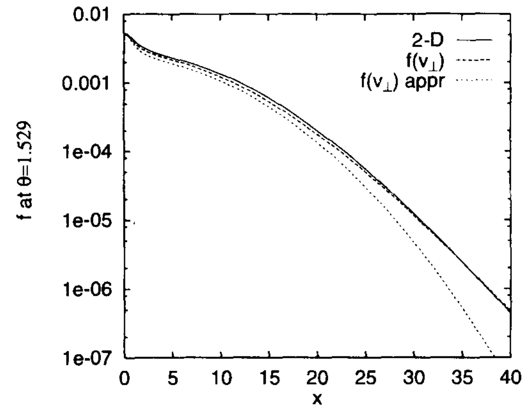

FIG. 7. $f(x,\pi/2)$ in steady-state from the solution on a 2-D grid compared with $\tilde{f}(v_\perp)$ [Eq. (12)], for $\langle P\rangle = 1.0$ MW/m^3. Also $\tilde{f}(v_\perp)_{appr}$ from the Stix explicit formula is given.

VII. CONCLUSIONS

In this paper we have presented an analysis of two-dimensional time-dependent solutions of the flux-surface-averaged Fokker–Planck equation for the distribution function of minority ions during ICRH, which was introduced by Stix in a classic paper.[1]

We have first applied the method of expansion in Legendre polynomials of the pitch-angle to the problem of deriving $f(v,\mu,t)$, in a manner similar to that by Stix, who had found the equations for the first two coefficients in the expansion. We have derived the equation for the generic coefficient $A_{2n}(v,t)$ of the expansion, for the case of heating at fundamental frequency and in the limit of a small Larmor radius, for an arbitrary n. We have then solved numerically the truncated systems of resulting equations for the coefficients of the expansion, starting from the case in which only the first term is kept (corresponding to the Stix pitch-angle-independent approach), and then successively increasing the number of retained terms up to $n_t=5$.

We have studied the convergence with increasing n_t of the sequence of the distribution function approximations $f^{(n_t)}$. Also, we have compared these approximations with the full numerical solution of the original equation obtained by means of an explicit finite difference method on a two-dimensional grid in velocity space, which we have developed. The convergence of the Legendre polynomial expansion to $f(v,\mu,t)$ has been shown to be very slow and non-uniform with respect to the speed v, i.e., for a fixed pitch-angle, the number of terms which are necessary to get a good approximation to f increases with increasing energy, the situation worsening as the pitch-angle θ varies from $\pi/2$ to 0. Therefore, in situations in which the distribution function at energies much higher than the thermal ones is needed, the method of expansion in the Legendre polynomials becomes impractical.

However, the convergence with increasing n_t of the sequences for the coefficients of the expansion is very good. In particular, a good approximation to the first expansion coefficient A_0, which represents the pitch-angle average of the distribution function, is obtained already with two terms kept in the expansion. This coefficient determines the angle independent moments of f, such as the total energy in the minority component, and the fusion reactivity between the tail minority ions and the bulk ones. When only the first term in the expansion is retained, the tail temperature of the pitch-angle average in the high energy region is underestimated by approximately 30%. Moreover, in high ICRH power density cases the accurate tail temperature of the pitch-angle average has been found to be 50% higher than that given by the Stix analytical formula $T_{tail}=T_e(1+\xi)$, as the latter formula contains both the inaccuracy associated with keeping only the first term in the expansion and that due to the use of approximating forms for the functions G and Φ which appear in the collisional diffusion coefficients.

We have also analysed Stix's $f(v_\perp)$ approximation. We have found that $f(v_\perp)$ in which G and Φ are calculated exactly is a very good approximation to $f(v,\theta=\pi/2)$ in steady-state for high ICRH power density cases. The Stix explicit analytical expression for $f(v_\perp)$ [Eq. (38) of Ref. 1] and the related formula for its tail temperature $T_{tail}=T_e(1+3\xi/2)$ have instead been found to be inaccurate, due to the use of the approximating forms for the functions G and Φ.

As far as the time dependence is concerned, our results show that a steady-state is achieved typically within 3 Spitzer slowing-down times for relaxation of fast ions on electrons, which for typical JET experiments is about 1.5 s. As the period of sawtooth oscillations is much shorter, we can conclude that the steady-state might not be reached in real experiments.

ACKNOWLEDGMENT

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