Convection without the Mixing Length Parameter

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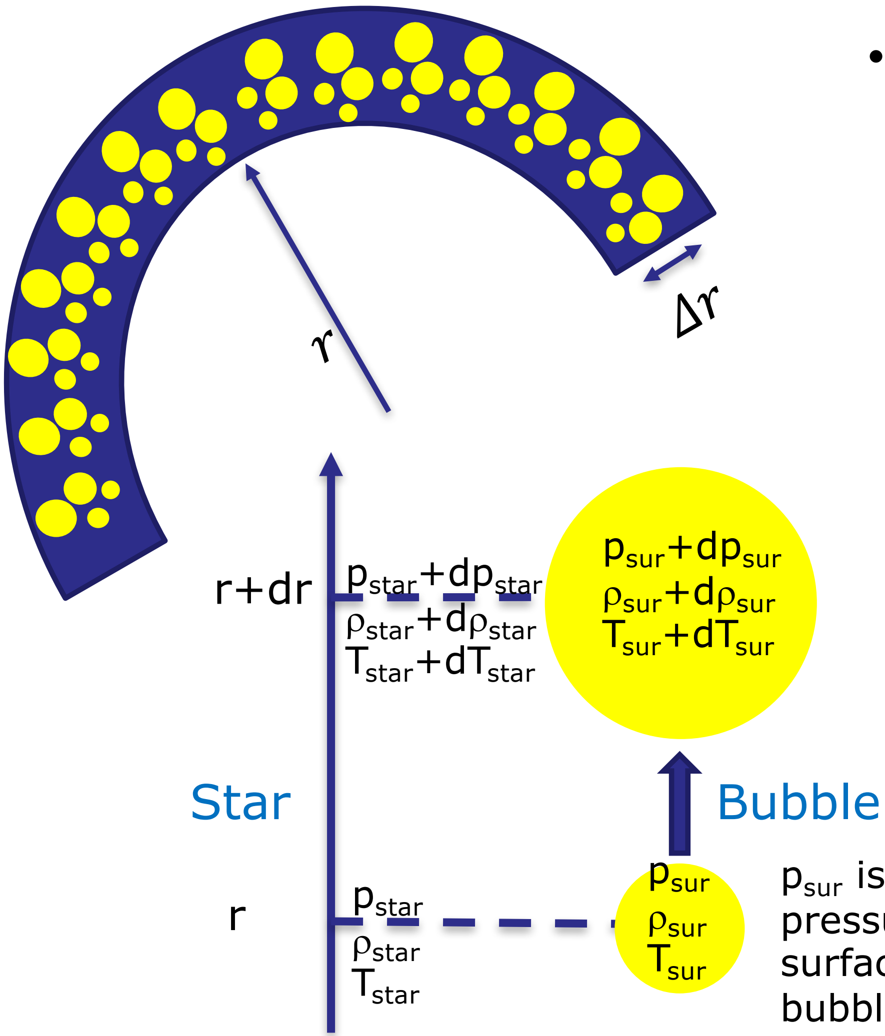
Rationale for replacing Mixing Length Theory

- The current approach for convection is Mixing Length Theory [Prandtl (1925), Böhm–Vitense (1958)]
- The universal applicability of the MLT is unproven and requires a calibration for each star
  $\Rightarrow$ a self-consistent theory will be a significant advance (and overdue)
- The correct treatment of convection is critical for stellar models throughout the H-R diagram
  $\Rightarrow$ affects every aspect of stellar and galactic evolution
- Advances in asteroseismology have allowed the internal structure of stars to be measured directly with increasing accuracy
  $\Rightarrow$ allows detailed confrontation with stellar models
- Advent of scale and accuracy of *Gaia* data requires stellar models of greater fidelity to fully utilise it
  e.g. location of red giant tracks depends sensitively on MLT parameter

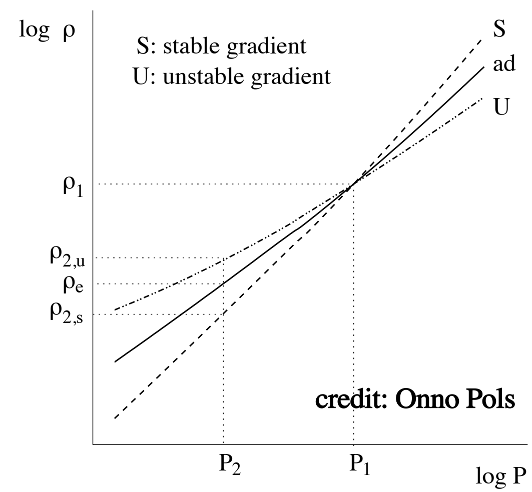
Convection Theory: stability criteria



p_{sur} is the pressure at the surface of the bubble

- Energy transfer by convection in the classical treatment is a linear "stability study" against non-spherical perturbations

Assuming that dr is small
and $p_{star}+dp_{star}= p_{sur}+dp_{sur}$
leads to the
Schwarzschild/Ledoux criterion
for instability
i.e. convection.



Mixing Length Theory

- The formulation is set in terms of:
  - φ_{rad} the radiative energy flux
  - φ_{cnv} the convective energy flux
  - ∇ the *stellar* temperature gradient with respect to pressure $\frac{d\log T}{d\log P}$
  - ∇_e the *element* temperature gradient with respect to pressure $\left|\frac{d\ln T}{d\ln P}\right|_e$
- With the assumption that $l_m \equiv \Lambda_m h_P$ where
  - l_m is the mean free path of a convective element
  - h_P is the distance scale of the pressure stratification
  - Λ_m is the proportionality constant (the Mixing Length Parameter)

  the system of equations

$$
\begin{aligned}
\varphi_{\mathrm{rad|cnd}} &= \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla \\
\varphi_{\mathrm{rad|cnd}} + \varphi_{\mathrm{cnv}} &= \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla_{\mathrm{rad}} \\
\bar{v}^2 &= g\delta\left(\nabla - \nabla_e\right)\frac{l_m^2}{8h_P} \\
\varphi_{\mathrm{cnv}} &= \rho c_P T\sqrt{g\delta}\,\frac{l_m^2}{4\sqrt{2}}h_P^{-3/2}\left(\nabla - \nabla_e\right)^{3/2} \\
\frac{\nabla_e - \nabla_{\mathrm{ad}}}{\nabla - \nabla_e} &= \frac{6acT^3}{\kappa\rho^2 c_P l_m \bar{v}},
\end{aligned}
$$

can be solved.

A self-consistent theory: two papers

- Pasetto et al (2014) MNRAS, <u>445</u>, 3592

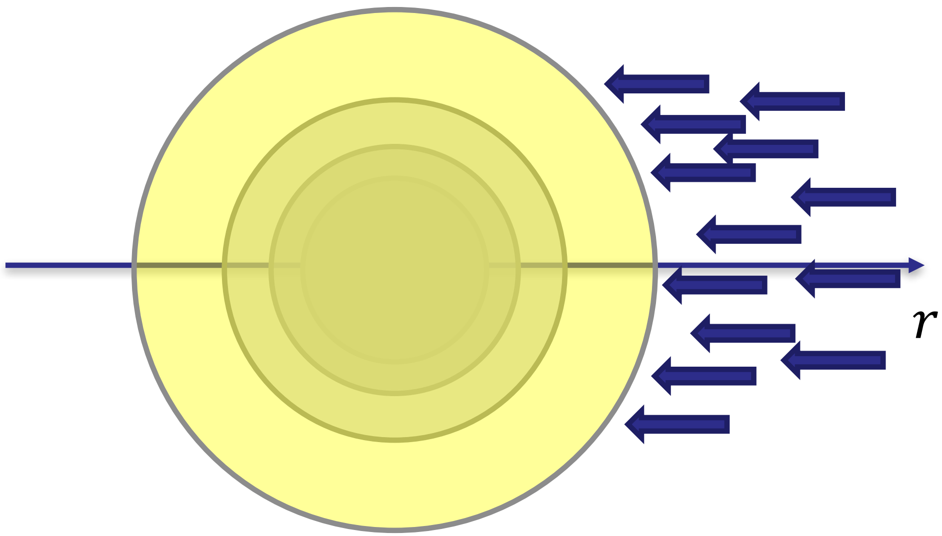
  - Paper 1 formulates the problem in the reference frame of the moving convective element
  - This allows the identification of a self-consistent additional constraint which can be used to close the system of equations without the external imposition of a mixing-length parameter
  - A comparison is made of the derived parameters (e.g., sound speed) in the Sun (where the Mixing Length Theory is calibrated)

- Pasetto et al (2016) MNRAS, <u>459</u>, 3182

  - Paper 2 presents the first stellar models using the non-MLT treatment
  - Evolutionary tracks are derived and compared to MLT-derived tracks
  - Derived internal parameters are compared between the two theories and agreement is found to be satisfactory

Self-consistent Theory: stability criterion

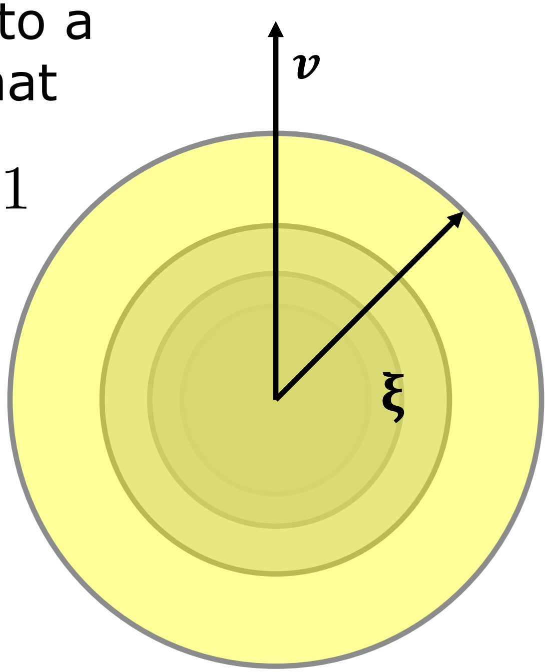
- The new treatment is in the co-moving frame of the bubble



co-moving coordinates
+
the concept of the
"velocity potential"

- The instability criterion now translates to a criterion that

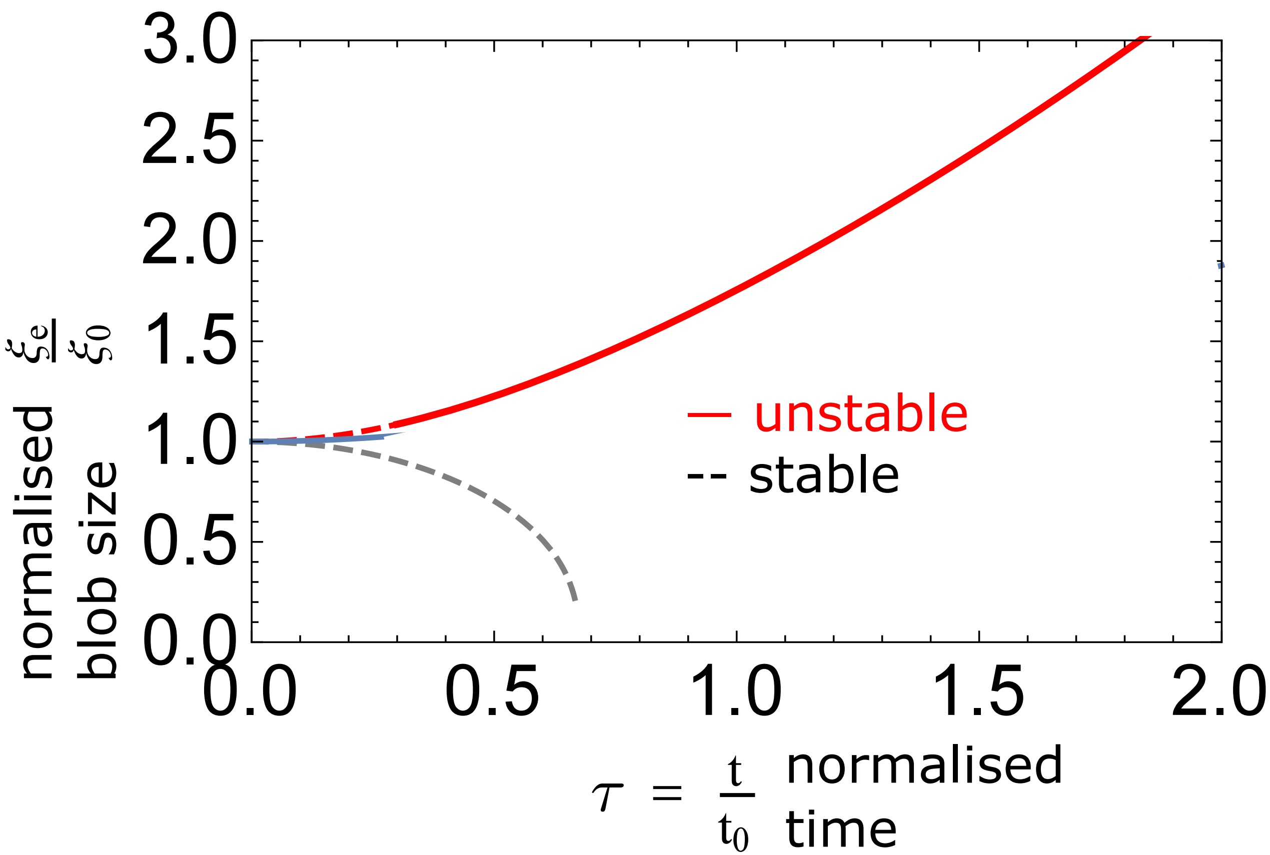
$$\frac{v}{\dot{\xi}_e} \ll 1$$



i.e. the new instability criterion is a <u>velocity</u> criterion that the expansion speed of the bubble is greater than the speed of the bubble in the star

Relation between blob size and time

- the unstable expansion is in terms of hyper-geometric functions which is <u>quadratic</u> in time in the leading term



Formulation

- Pasetto et al (2014) derives 6 equations in 6 unknowns:

$$
\begin{aligned}
\varphi_{\mathrm{rad/cnd}} &= \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla \\
\varphi_{\mathrm{rad/cnd}} + \varphi_{\mathrm{cnv}} &= \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla_{\mathrm{rad}} \\
\bar{v}^2 &= \frac{\nabla - \nabla_e - \frac{\varphi}{\delta}\nabla_\mu}{\frac{3h_P}{2\delta\bar{v}\tau} + \left(\nabla_e + 2\nabla - \frac{\varphi}{2\delta}\nabla_\mu\right)}\bar{\xi}_e g \\
\varphi_{\mathrm{cnv}} &= \rho c_P T \left(\nabla - \nabla_e\right)\frac{\bar{v}^2\tau}{h_P} \\
\frac{\nabla_e - \nabla_{\mathrm{ad}}}{\nabla - \nabla_e} &= \frac{4acT^3}{\kappa\rho^2 c_P}\frac{\tau}{\bar{\xi}_e^2} \\
\bar{\xi}_e &= \frac{g}{4}\frac{\nabla - \nabla_e - \frac{\varphi}{\delta}\nabla_\mu}{\frac{3h_P}{2\delta\bar{v}\tau} + \left(\nabla_e + 2\nabla - \frac{\varphi}{2\delta}\nabla_\mu\right)}\bar{\chi},
\end{aligned}
$$

- The two new unknowns are:

$\bar{\xi}_e$ the mean size of the convective element and
$\bar{v}$ the mean velocity

Solving the system of equations

- After substitutions and definition of new variables, the 6 equations reduce to the following:

$$\frac{Y^2}{(W-\eta)\,(\eta-Y)} = \frac{1}{3}\,\frac{\bar{\chi}}{\tau^2}$$

where:

$$W \equiv \nabla_{\rm rad} - \nabla_{\rm ad} > 0,$$

$$\eta \equiv \nabla - \nabla_{\rm ad},$$

$$Y \equiv \nabla - \nabla_e$$

$$\chi \equiv \frac{\xi_e}{\xi_0}$$

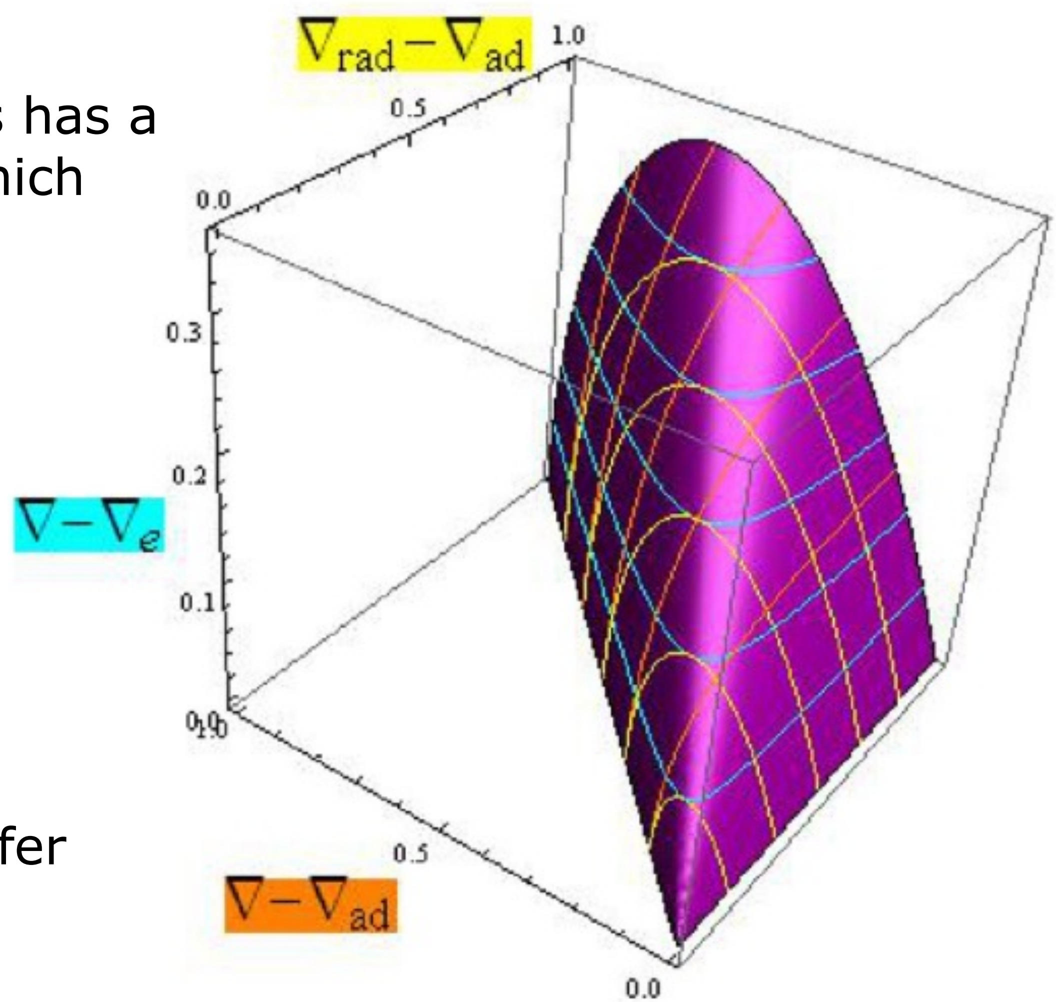
- but, recall $\chi \equiv \frac{\xi_e}{\xi_0} \propto \tau^2$ from previous graph, so $\frac{\bar{\chi}}{\tau^2} =$ constant

Outcome of the reduction of dimensionality

- The new system of equations has a new invariant manifold on which all the solutions live

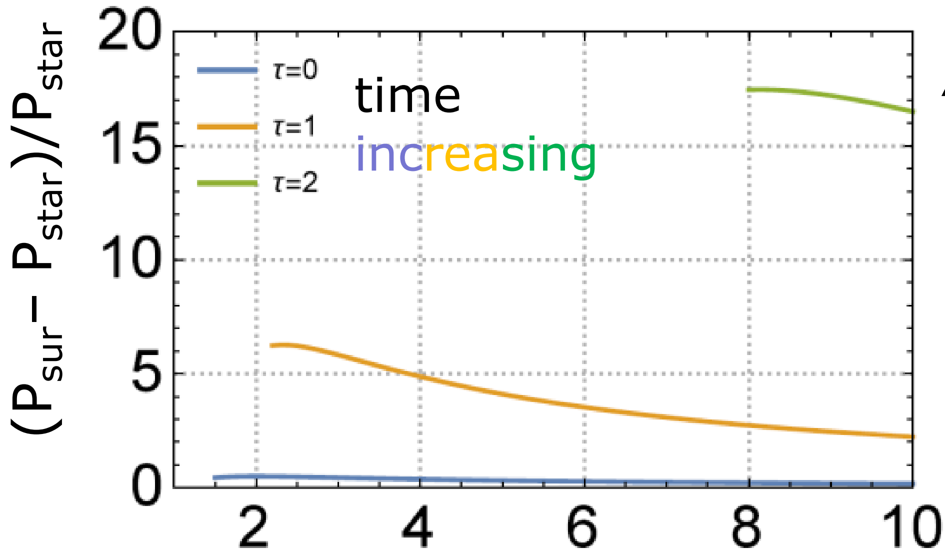
- The temperature gradients
  at each point
  in any star
  are located on this manifold

- "Theorem of Unicity”:
  a relation between the
  4 fundamental gradients
  that govern the energy transfer
  inside a star.



Another important consequence

- The treatment leads to a non-hydrostatic equilibrium theory, hence non-hydrostatic equilibrium models of atmospheres

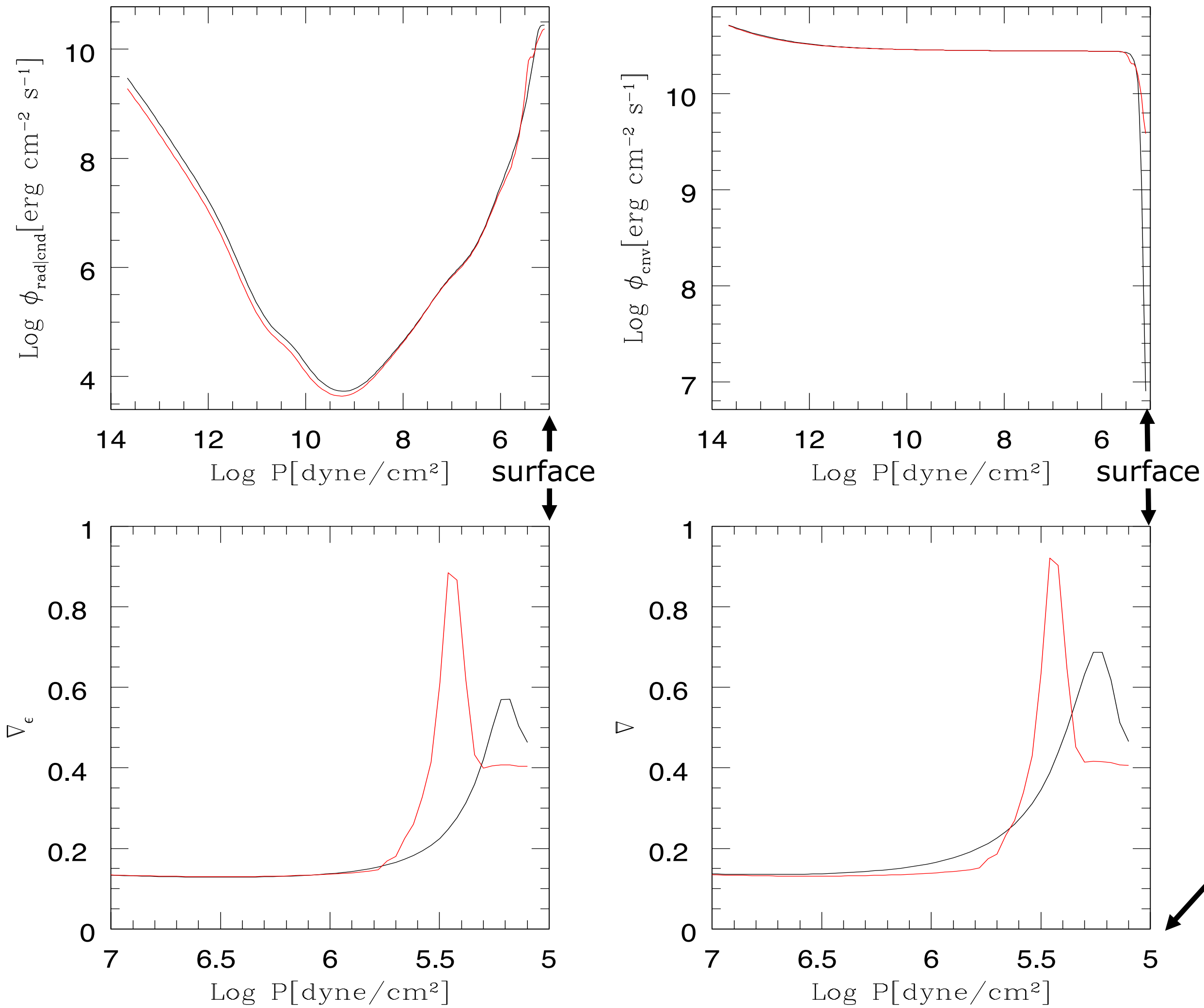


Y axis is the pressure difference across the blob interface compared to pressure in the same stellar layer far from the blob

$\chi \equiv \frac{\xi_e}{\xi_0}$ normalised blob size

- This is a fundamental advance on the MLT where equilibrium is assumed to be reached at the end of the bubble movement

Results (1): outer convective layers



Solar Model

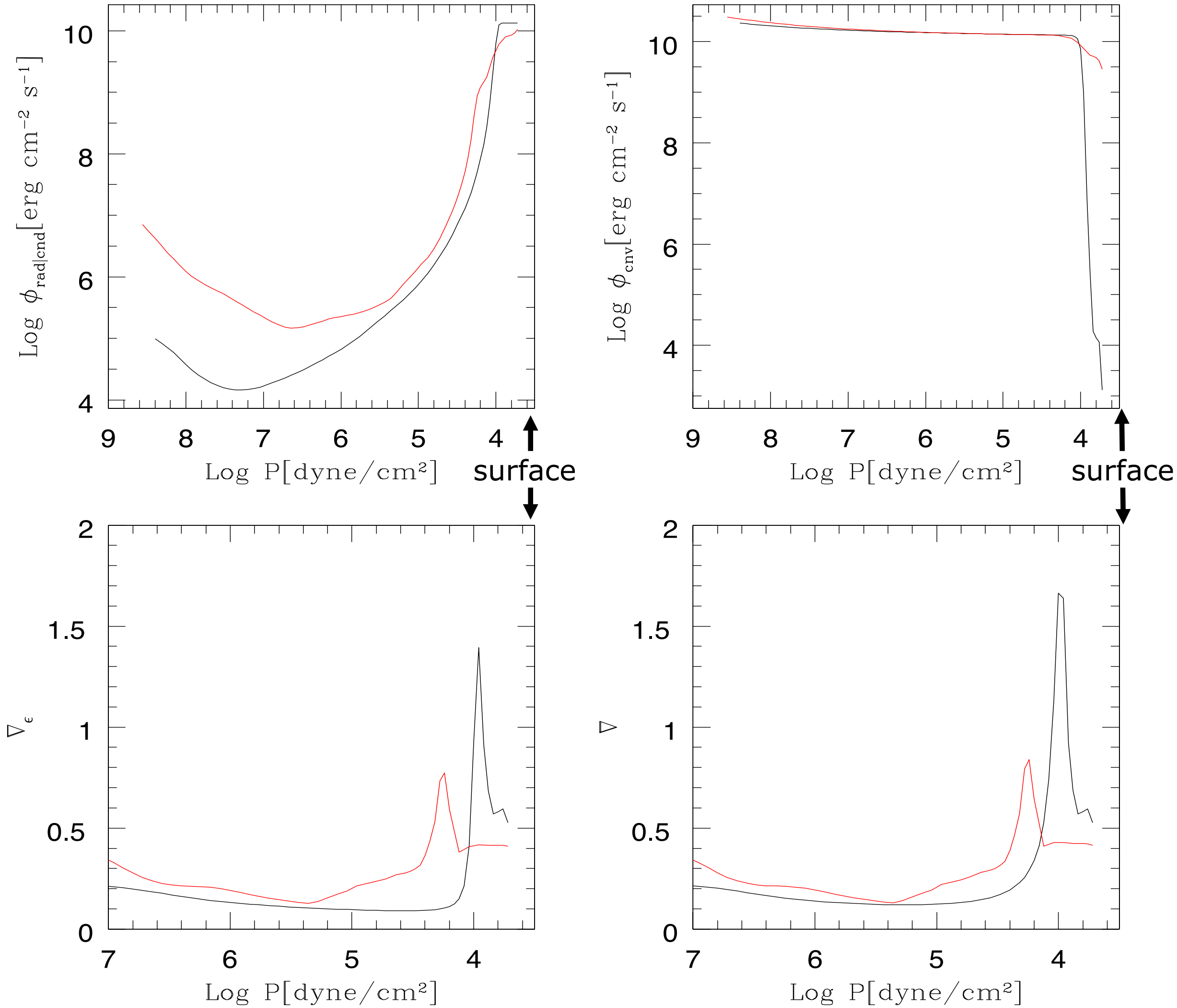
Bertelli et al. (2008)

black: MLT $(\Lambda = 1.68)$

red: this work

expanded pressure scale

Results (2): outer convective layers



2M$_\odot$ RGB star

log $L/L_\odot$ =2.598, log T_{eff} =3.593

Bertelli et al. (2008)

black: MLT ($\Lambda = 1.68$)

red: this work

Outer convective layers: comparison

- For Solar model:
  - good agreement for convective and radiative fluxes throughout
  - temperature gradients are in good agreement except for surface layers

    Reason: treatment incomplete at the boundary

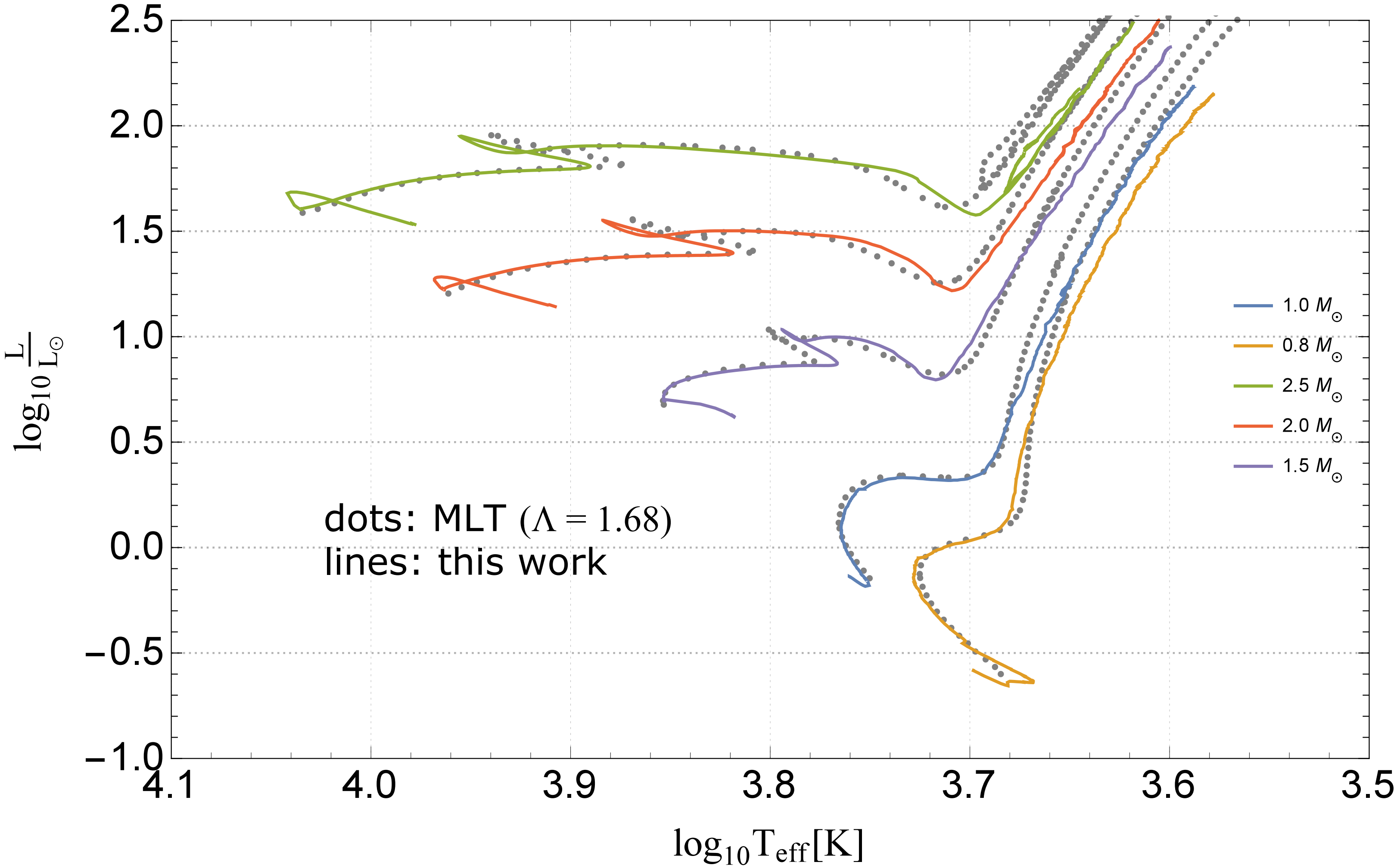
- For $2M_{\odot}$ model:
  - good agreement for convective fluxes
  - divergence to lower boundary for radiative fluxes

    Reason: these solutions are not constrained to match the inner solution at the transition layer
  - temperature gradients as for Solar model

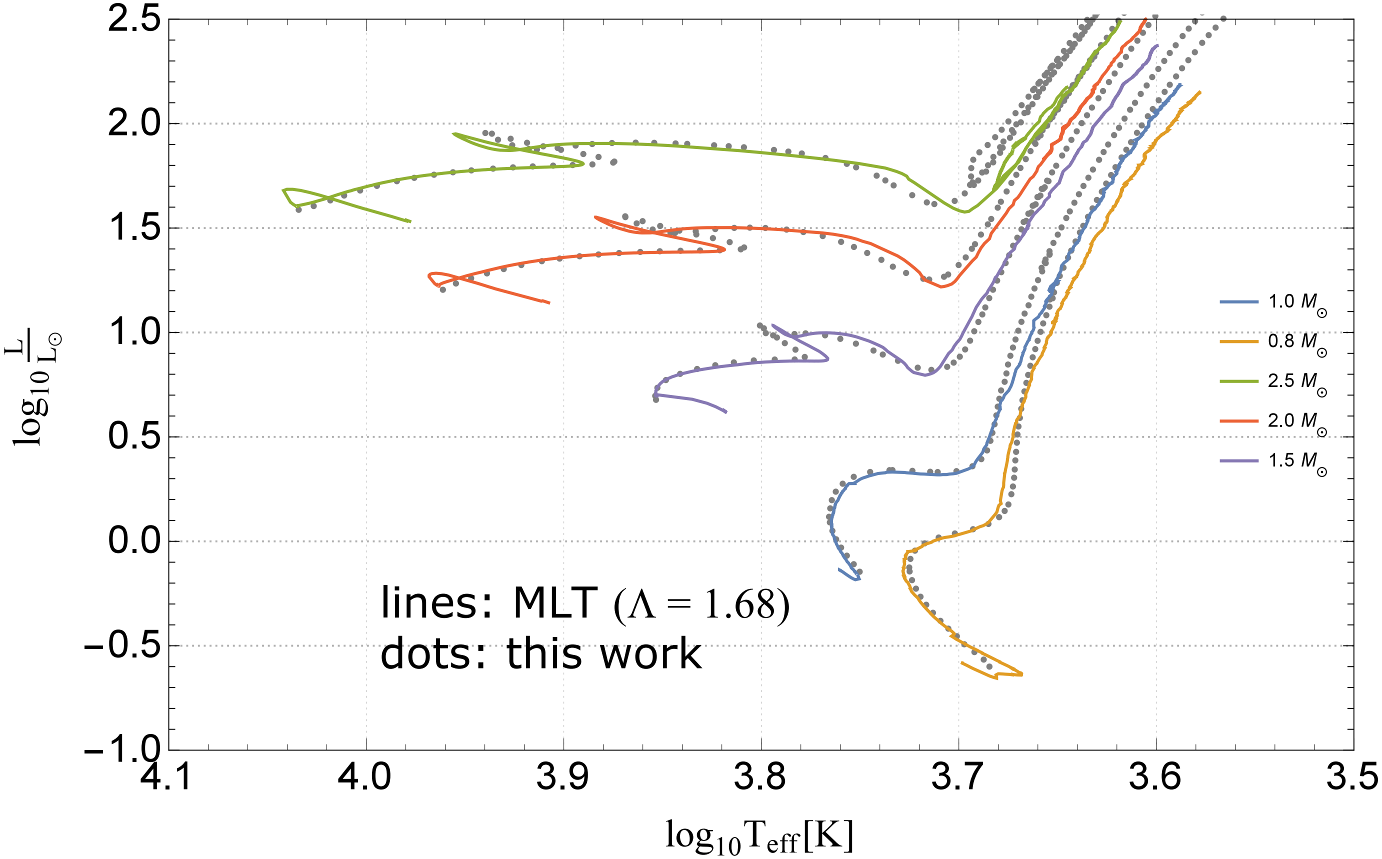
- To constrain the inner solution, need to calculate full stellar models

- For these full calculations, Mixing Length Theory used for the interiors

Results 3: Stellar models



Results 3: Stellar models



Note: away from the well-calibrated cases, care should be exercised in which approach is the considered to be the reference.

Full stellar models: Overshooting

- The new theory does not yet include overshooting
- However, it derives the acceleration acquired by convective elements
  under the action of the buoyancy force
  in presence of the inertia of the displaced fluid and gravity.
- Therefore, it is also best suited to describe convective overshooting
- Extension of the atmospheric modelling to include overshooting is in preparation (Pasetto et al 2017)

Summary

- The correct treatment of convection is critical for stellar models throughout the H-R diagram
- The current standard approach using Mixing Length Theory requires
  - an additional relation not justified within the theory
  - with a calibration which is not universal
- A self consistent theory has been derived which allows the system of equations to be closed – this depends on
  - a formulation within co-moving coordinates
  - a new definition of the stability criterion
  - an identification of a growth-rate relation which allows the elimination of one of the variables in the formulation
- The new theory agrees closely with the MLT in the case of the Sun where the MLT is well-calibrated
- The new theory predicts sensible stellar evolutionary tracks, which may already be better than MLT outside where this is calibrated.
- The new theory can be extended to be applied broadly (geology, meteorology, oceanography) with the addition of viscosity terms